RELATIVISTIC AND NON-RELATIVISTIC MOMENTUM SPACE WAVEFUNCTIONS FOR CHARMONIUM AND BOTTOMONIUM STATES

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Abstract

In the constituent quark model, charmonium and bottomonium are considered to be the bound states of charm/anti-charm and bottom/antibottom quarks. We use the linear potential to represent the long distance confining part of the potential and a Coulomb like potential to represent the short distance one-gluon exchange part of the potential. In order to study the general features of the wave functions, it is not necessary to include spin dependent parts in the potential. Using the above mentioned potentials, we solved the Schrodinger equation with non-relativistic kinematics and also with relativistic kinematics. We solve these equations by expanding the momentum space wavefunction in a complete set of orthonormal basis functions and turning the Schrodinger equation into a standard matrix eigen-value equation. We vary the masses of the quarks, and the strengths of the potentials until we get a satisfactory fit to the spin averaged mass spectra of the desired $q\bar{q}$ system. The eigenvectors gives the coefficients of the linear combination in the wavefunction expansion from which we can construct the wavefunctions. We compare the non- relativistic and relativistic wavefunctions for each state in $b\bar{b}$ and $c\bar{c}$ systems.

Keywords: quark, linear potential, momentum space

Introduction

Meson as a two-body bound state of quark-antiquark pair has been a fertile ground for the study of two-body relativistic equations and also for phenomenology. Most studies restrict themselves to non-relativistic Schrodinger equation in position space, since the solution methods are well known and in some cases, analytical solutions exist. Commonly used methods are the well known Numerov method combined with wavefunction and its derivative matching at a predetermined location. The input is the eigen-energy and it is varied until the desired tolerance is achieved for the log-derivative matching. For each angular momentum value *l* this has to be done for all the states of interest. Although this is not an economical way of doing things, it can be done. But if we want to use relativistic kinematics, one can no longer use position space representation of the Schrodinger equation.

The non-relativistic kinetic energy operator $\hat{p}^2/2\mu$ (μ is the reduced mass)is now replaced by $\sqrt{\hat{p}^2 + m_1^2}$+$\sqrt{\hat{p}^2 + m_2^2}$ in the center of mass frame. Since quantization gives $\hat{p} \rightarrow -i\hbar\nabla$, we will have differential operator under the radical sign. This problem is best treated in momentum representation. In this case both the non-relativistic and the relativistic Schrodinger equations are integral equations and the momentum operator $\hat{p}$ becomes just a number in this representation. In the case of linear potential and Coulomb like potentials, there is a minor complication of removable singularities in the momentum representation of these potentials. Well known subtraction methods exist. We can also add spin-spin and spin-orbit potentials to our problem without any technical complications. But the spin-orbit splitting between $^1S_0$ and $^3S_1$ in $b\bar{b}$ system is about 50 MeV while the mass of the meson is about10,000 MeV. This is of the order of 0.5%. Therefore, in this stage of the study of the wavefunctions, we neglect the spin dependent part of the potential. In the next section, we will describe how we will solve the Schrodinger equation in momentum space with non-relativistic and relativistic kinematics. We will also explain how the wave function can be obtained once the eigen-value problem is solved. In section III, we will show our results of eigen-

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values and their fit to the spin-averaged meson spectra. We will also compare the relativistic and non-relativistic wavefunctions. We than make conclusions based on our results.

The Solution Method

The Schrodinger equation in momentum representation can be written as

$$D(m_1, m_2, p) + \int V(\bar{p}, \bar{p}')\phi(\bar{p}')d\bar{p}' = E\phi(\bar{p}) \qquad (1)$$

Here $D(m_1, m_2, p) = p^2/2\mu$ for the case with non-relativistic kinematics and $D(m_1, m_2, p) = \sqrt{m_1^2 + p^2} + \sqrt{m_2^2 + p^2}$ for the relativistic case. Now $V(\bar{p}, \bar{p}')$ is the Fourier transform of the position space potential.

$$V(r) = \lim_{\eta \to 0} \left(\sigma r e^{-\eta r} - \frac{C e^{-\eta r}}{r} \right) \qquad (2)$$

Here $\sigma$and C are the strengths of the linear and the Coulomb like potentials. We have used the exponential damping factor η since the straight forward Fourier transforms of linear potential and Coulomb potential do not exist. After the Fourier transform and partial wave decomposition, we can take the $\eta \to 0$ limit explicitly .

The Fourier transforms of the damped potentials in equation (2) are given by

$$V(\bar{p}, \bar{p}') = \lim_{\eta \to 0} \left[\frac{\sigma}{2\pi^2} \frac{\partial^2}{\partial \eta^2} \left(\frac{1}{q^2 + \eta^2} \right) - \frac{C}{2\pi^2} \left(\frac{1}{q^2 + \eta^2} \right) \right] \qquad (3)$$

Since we do not have any potential that couples angular momentum, we can take the orbital angular momentum l as a good quantum number and write

$$\phi(\bar{p}) = \phi_{nl}(p) \mathrm{Y}_l^{\mathrm{m}}(\hat{p}) \qquad (4)$$

Now after the angular decomposition Schrodinger equation can be written for each l as

$$D(m_1, m_2, p^2)\phi_{nl}(p) + \int_0^\infty V_l(p, p')\phi_{nl}(p')p'^2 \, dp' = E_{nl}\phi_{nl}(p) \qquad (5)$$

Here the l^{th} partial wave component of the potential is given by

$$V_l(p, p') = 2\pi \int_{-1}^{1} V(p, p')P_l(x)dx \qquad (6)$$

$$= \frac{1}{\pi p p'} \lim_{\eta \to 0} \left(\sigma \frac{\partial^2}{\partial \eta^2} Q_l(y) - C Q_l(y) \right) \qquad (7)$$

Here, $Q_l(y)$ is the Legendre polynomial of the second kind and the argument $y = (p^2 + p'^2 + \eta^2)/2pp'$. In the $lim_{\eta \to 0}$limit, $Q_l(y)$ and its derivatives have a removable singularity as can be seen here.

$$Q_l(y) = Q_0(y)P_l(y) - W_{l-1}(y) \qquad (8)$$

and $Q_0(y)$ and $W_{l\text{-}1}(y)$ are given by

$$Q_0(y) = \frac{1}{2} \ln \left[\frac{\mathrm{y}+1}{\mathrm{y}-1} \right] \qquad (9)$$

and

$$W_{l-1}(y) = \sum_{\mathrm{m}=1}^{\mathrm{l}} \frac{1}{\mathrm{m}} \mathrm{P}_{l-\mathrm{m}}(\mathrm{y}) \, \mathrm{P}_{\mathrm{m}-1}(\mathrm{y}) \qquad (10)$$

We immediately see that although $W_{l\text{-}1}(y)$ has no singularties but $Q_l(y)$ and its derivatives do through $Q_0(y)$ at $y = 1$ which corresponds to $p=p'$ case in the limit of $\eta \to 0$. In order to handle these singularities, we use the method developed by Maung, Kahana and Norbury [2]. In order to

solve equation (5), we expand the wavefunction $\phi_{nl}(p)$ in complete orthonormal basis set of functions. i.e

$$\phi_{nl}(p) = \sum_{\alpha=1}^{N} C_{\alpha l} g_{\alpha l}(p) \qquad (11)$$

Here, for the $g_{\alpha l}(p)$'s we use a complete orthonormal set of functions given in terms of Jacobi polynomials [4] and $C_{\alpha l}$ are the expansion coefficients. The set of functions $g_{\alpha l}(p)$ are given by

$$g_{\alpha l}(p) = \frac{1}{\sqrt{N_{\alpha l}}} \frac{(p/b)^l}{[(p/b)^2+1]^{l+2}} P_{\alpha}^{\left(l+\frac{3}{2},l+\frac{1}{2}\right)}\left(\frac{p^2-b^2}{p^2+b^2}\right) \qquad (12)$$

$$N_{\alpha l} = \frac{b^3}{2(2n+2l+3)} \frac{\Gamma(n+l+5/2)\Gamma(n+l+3/2)}{n!\Gamma(n+2l+3)} \qquad (13)$$

where $N_{\alpha l}$ is the normalization constant and b is a parameter that be used as the variational parameter. $P_{\alpha}^{(a,c)}(x)$ are the Jacobi polynomials. These expansion functions $g_{\alpha l}(p)$ obey orthonormal condition

$$\int_0^{\infty} g_{\alpha l}(p) g_{\beta l}(p) p^2 \, dp = \delta_{\alpha\beta} \qquad (14)$$

We now use the expansion given by equation (11) in (5) and furthermore, we multiply by $p^2 g_{\alpha l}(p)$ and integrating over p we obtain

$$\sum_{\alpha=1}^{N} C_{\alpha l}\left(\int_0^{\infty} g_{\beta l}(p) D(m_1, m_2, p^2) g_{\alpha l}(p) p^4 dp\right) + \iint_0^{\infty} g_{\beta l}(p) V_l(p,p') g_{\alpha l}(p') p^2 p'^2 dp dp'\Big) = E_{\alpha l} C_{\beta l} \qquad (15)$$

This is in the form of a simple matrix eigenvalue equation for each l. i.e.

$$\sum_{\alpha=1}^{N} A_{\beta\alpha} C_{\alpha l} = E_{\alpha l} C_{\beta l} \qquad (16)$$

Once the eigen-equation is solved, we obtain the eigen-energies and for each energy we get a set of coefficients C_{α}.Then, by using equation (11) we can construct the wavefunction corresponding to the desired state.

Results

In this section we show the results of our calculations. As explained previously, we do not include spin-spin and spin-orbit potentials. Therefore, before we fit the meson spectrum, we do the spin-averaging of the masses. First of all, meson states are given in the spectroscopic notation as $n\,{}^{2S+1}L_J$, where the S is in the superscript is the total spin which is either $S = 0$ (singlet state) or $S = 1$ (triplet state). L is the orbital angular momentum quantum number and they are traditionally named as S, P, D etc. for $L = 0, 1, 2, \ldots$ respectively. For example, for $l = 0$ case there are four states, namely ${}^1S_0$ (1 state) and ${}^3S_1$ (3 states). Therefore, the spin-average masses $M(nS)$ for $l = 0$ states are calculated from

$$M(nS) = \frac{[M(n\,{}^1S_0) + 3M(n\,{}^3S_1)]}{4} \qquad (17)$$

and the spin-averaged masses for P-states are calculated by

$$M(nP) = \frac{M(n\,{}^3P_0) + 3M(n\,{}^3P_1) + 3M(n\,{}^1P_1) + 5M(n\,{}^3P_2)}{12} \qquad (18)$$

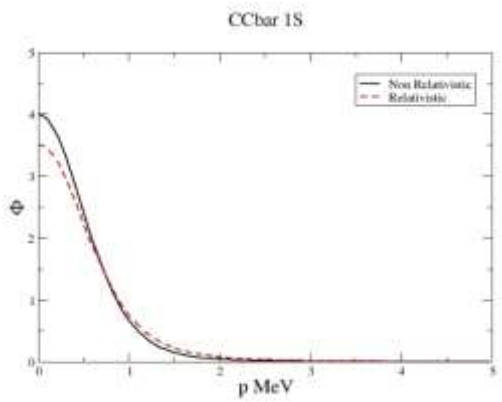
Table 1 Spin-averaged experimental meson masses(MeV) and calculations (Non-Relativistic) M_c **= 1321.5** MeV**,** M_b **= 4763.8** MeV

State	$C\bar{C}$ Calculation	$C\bar{C}$ Experiment	$B\bar{B}$ Calculation	$B\bar{B}$ Experiment
1S	3068.84	3068.65	9444.58	9444.98
2S	3720.06	3673.95	10036.25	10017.2
3S	4206.86		10400.47	
4S	4556.18		10696.73	
5S	5000.25		10957.12	
1P	3518.43	3525.31	9928.74	9899.73
2P	4029.52		10303.95	
3P	4462.01		10607.15	
4P	4848.62		10872.57	
5P	5203.97		11113.65	

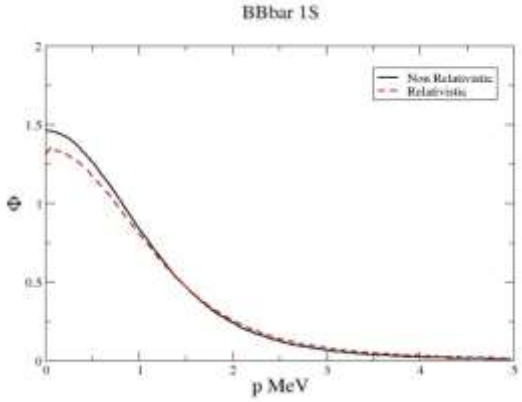
Table 2 Spin-averaged experimental meson masses(MeV) and calculations (Relativistic) M_c **= 1352.19** MeV**,** M_b **= 4783.42** MeV

State	$C\bar{C}$ Calculation	$C\bar{C}$ Experiment	$B\bar{B}$ Calculation	$B\bar{B}$ Experiment
1S	3068.47	3068.65	9444.50	9444.98
2S	3679.80	3673.95	10044.11	10017.2
3S	4123.95		10404.81	
4S	4496.53		10695.39	
5S	4826.01		10949.15	
1P	3480.28	3525.31	99564.39	9899.73
2P	3945.98		10324.25	
3P	4332.98		10620.19	
4P	4673.12		10878.08	
5P	4981.049		11326.93	

In these calculations, we fixed the strength of the linear potential to 0.2 $GeV^2$and Coulomb like potential strength to -0.5. We vary only the quark masses. Once the 1*S* state is adjusted to the experimental value with a tolerance of 0.05% we keep the quark mass value. We then use the same parameter set for L=1 p-wave case. We do the same procedure for both the non relativistic and relativistic cases.



(a)



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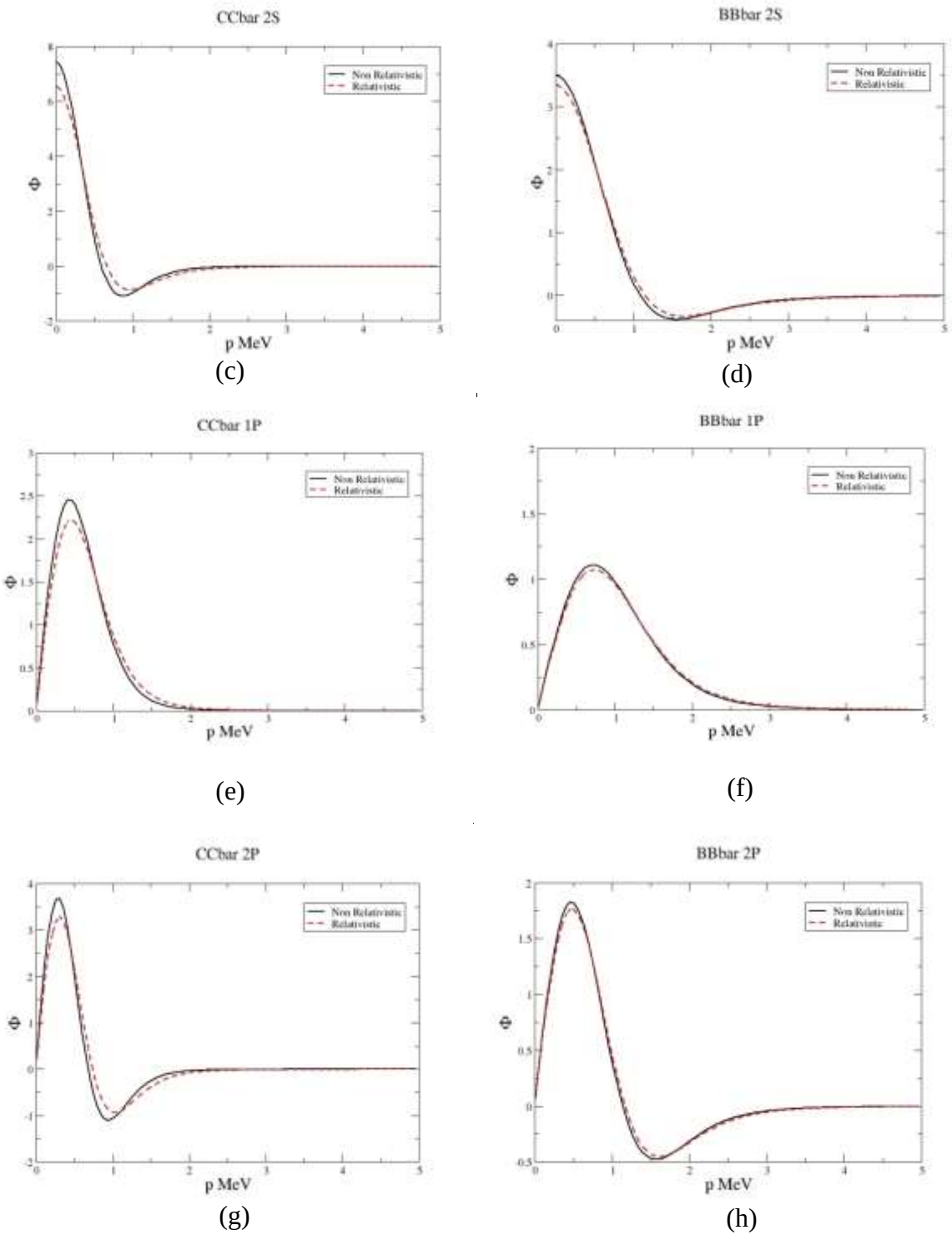

Figure (a-h) Non-relativistic and relativistic wavefunctions for b-bbar and c-cbar syatems

Conclusions

In the figures, we compare non-relativistic and relativistic wavefunctions for, 1*S*, 2*S*, 1*P* and 2*P* states for b-bbar and c-cbar systems. All wavefunctions are normalized. First of all we note that just like in the r-space wavefunctions, the ground state for each angular momentum *L* has no nodes in the wavefunction. The first excited state has one node etc. We see that the differences

between the relativistic and non-relativistic wavefunctions are small for b-bbar system. It is reasonable since it is a much heavier system. We also note that as *L* becomes higher, the difference is more pronounced and for the same *L*, excited states show more difference between relativistic and non-relativistic case. In the future, the prediction of decay rates will be done and compared with experimental results. In these calculations, the differences in the wavefunction will be important.

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